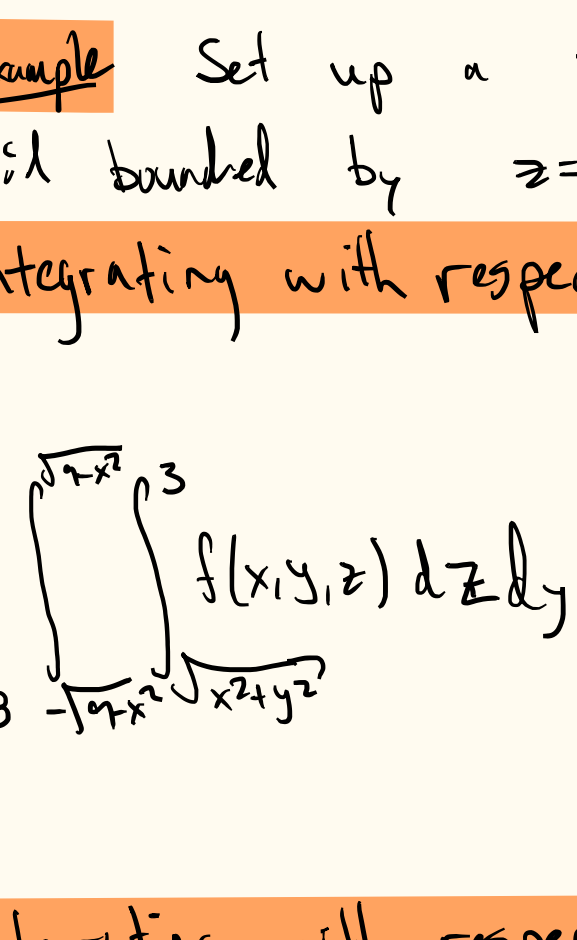


Section 11.7.1

Triple Integrals

Recollection: We can set up iterated triple integrals over "nice" solid regions R .

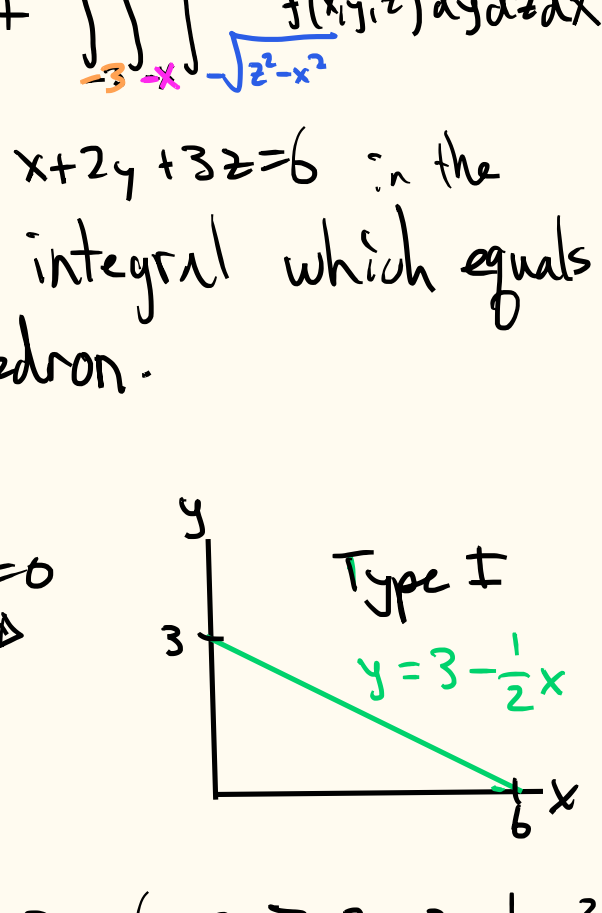


$$\int_a^b \int_{g_1(x)}^{g_2(x)} \int_{h_1(x,y)}^{h_2(x,y)} f(x,y,z) dz dy dx$$

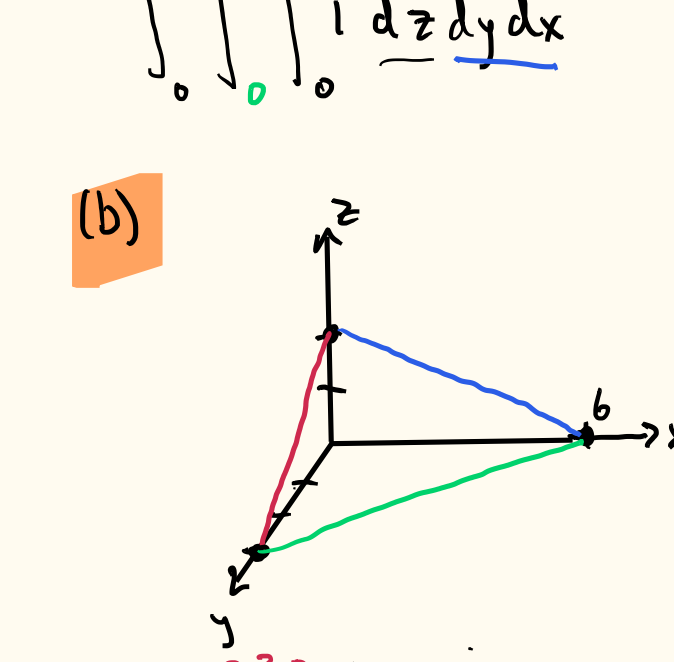
Example Set up a triple integral over the solid bounded by $z = \sqrt{x^2 + y^2}$ and $z = 3$.

Integrating with respect to z first:

$$\int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{\sqrt{x^2+y^2}}^3 f(x,y,z) dz dy dx$$



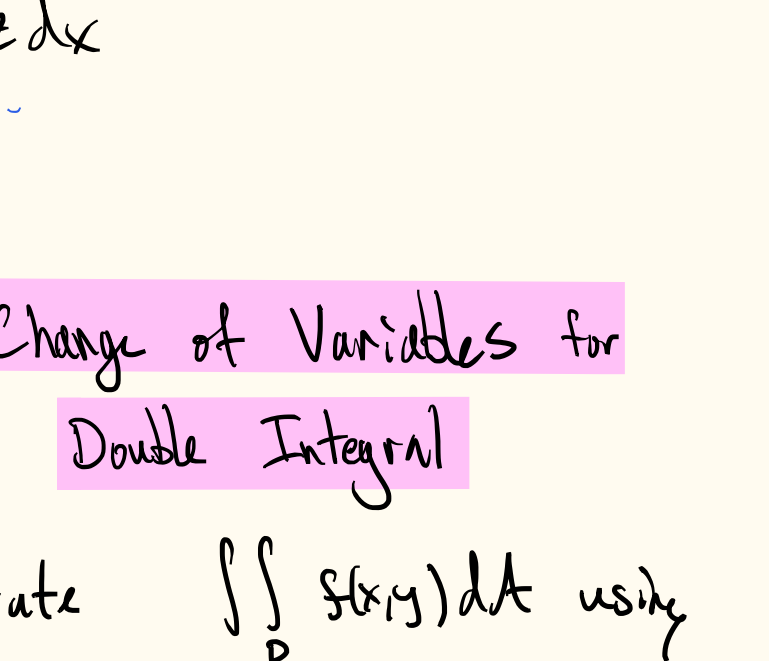
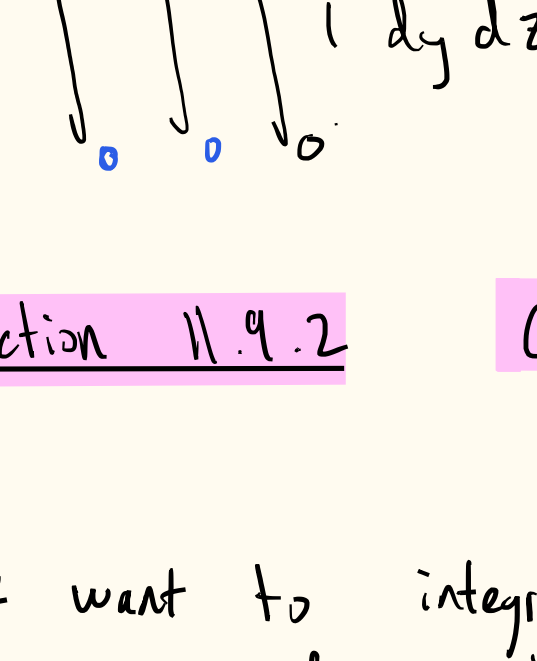
Integrating with respect to y first:



$$= \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{\sqrt{x^2+y^2}}^3 f(x,y,z) dz dy dx + \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f(x,y,z) dz dy dx$$

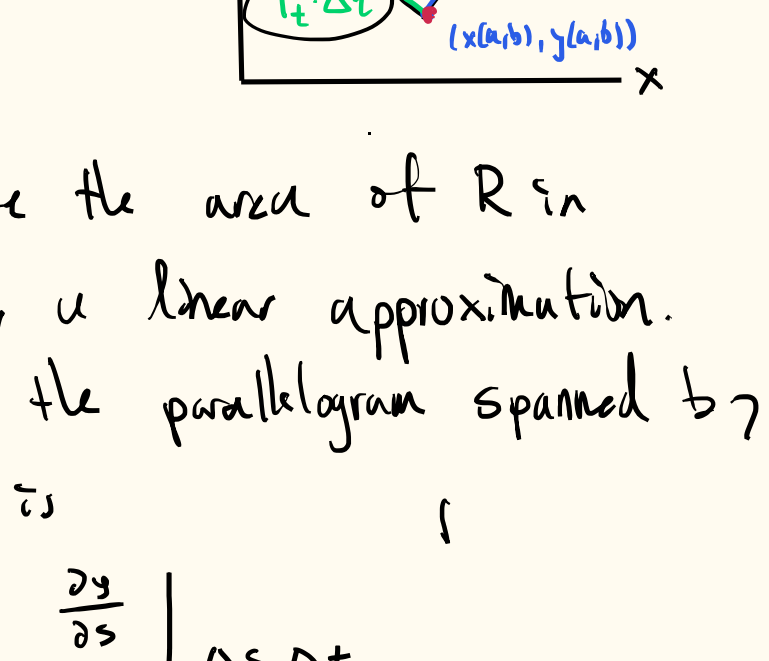
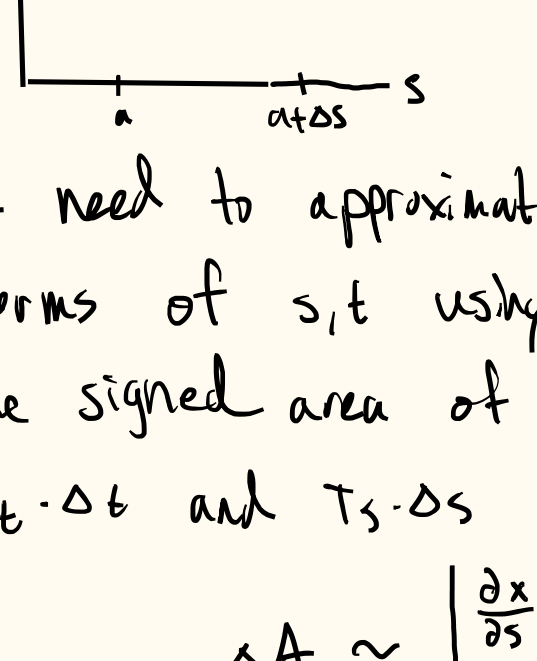
Example Solid bounded by $x + 2y + 3z = 6$ in the first octant. Set up an integral which equals the volume of the tetrahedron.

(a) $dz dy dx$ first.



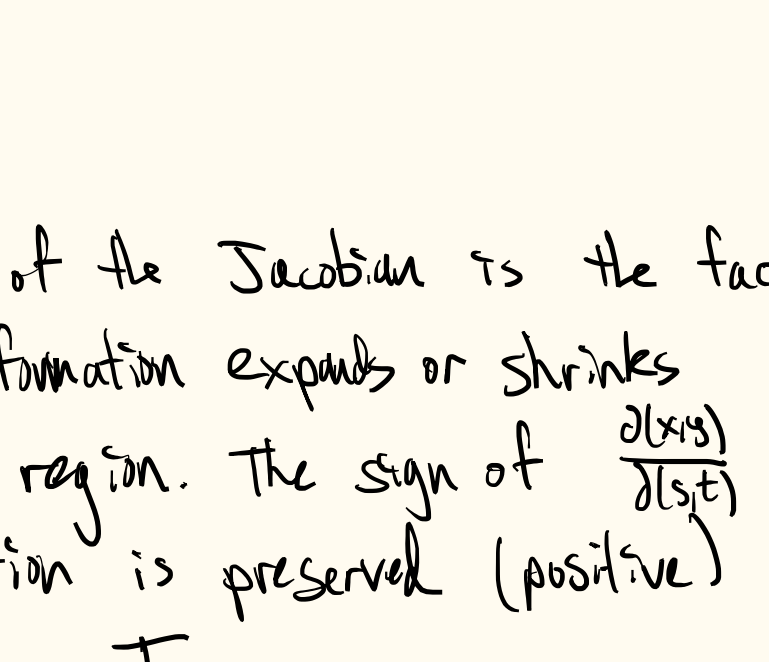
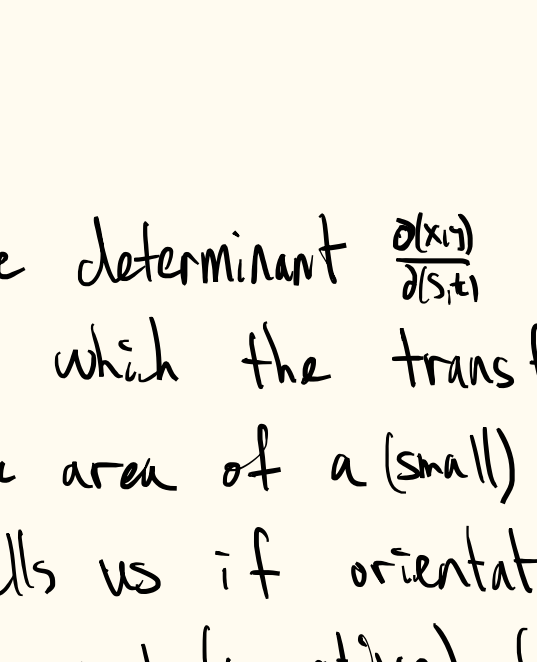
$$\int_0^6 \int_0^{3-\frac{1}{2}x} \int_0^{2-\frac{1}{3}x-\frac{2}{3}y} 1 dz dy dx$$

(b) $dx dy dz$ first.



$$\int_0^2 \int_0^{3-\frac{2}{3}z} \int_0^{6-3z-2y} 1 dx dy dz$$

(c) $dy dz dx$ first.



$$\int_0^6 \int_0^{2-\frac{1}{3}x} \int_0^{3-\frac{2}{3}z-\frac{1}{2}x} 1 dy dz dx$$

Section 11.9.2

Change of Variables for Double Integral

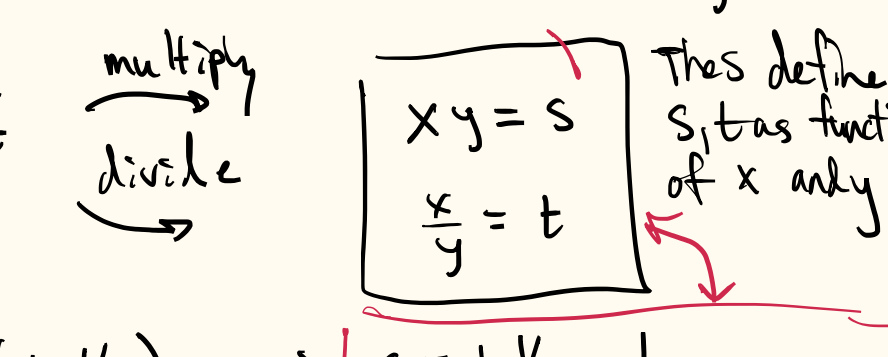
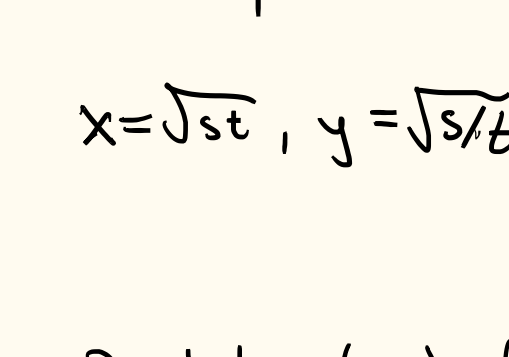
We want to integrate $\iint_R f(x,y) dA$ using a change of variable

$$x = x(s,t) \quad y = y(s,t)$$

this defines a transformation $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$

$$T(s,t) = (x(s,t), y(s,t))$$

which transforms regions in s,t -plane into regions in the xy -plane.



We need to approximate the area of R in terms of s,t using a linear approximation. The signed area of the parallelogram spanned by $T_t \cdot \Delta t$ and $T_s \cdot \Delta s$ is

$$\Delta A \approx \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix} \Delta s \Delta t$$

$$\text{So } dA = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix} ds dt$$

The matrix $\begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix}$ is called the Jacobian.

The determinant of the Jacobian is denoted

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix}$$

The determinant $\frac{\partial(x,y)}{\partial(s,t)}$ of the Jacobian is the factor by which the transformation expands or shrinks the area of a (small) region. The sign of $\frac{\partial(x,y)}{\partial(s,t)}$ tells us if orientation is preserved (positive) or not (negative) by T .

Change of Variables (Double Integral)

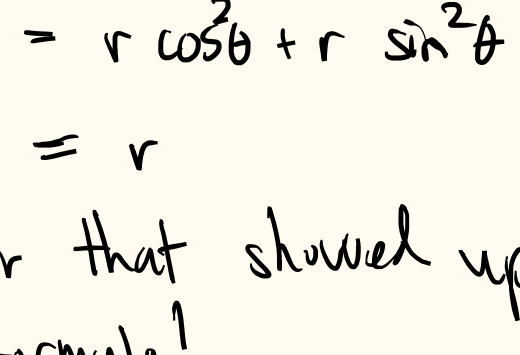
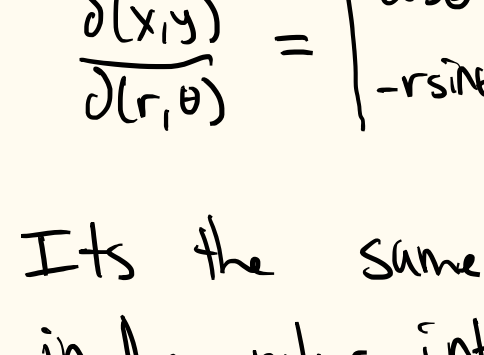
Suppose $x(s,t), y(s,t)$ transforms a closed and bounded region R' in s,t -coord into a closed and bounded region R in xy -coord.

then,

$$\iint_R f(x,y) dA = \iint_{R'} f(x(s,t), y(s,t)) \left| \frac{\partial(x,y)}{\partial(s,t)} \right| ds dt$$

Since the orientation of R' and R could be different, we have to use the absolute value $\left| \frac{\partial(x,y)}{\partial(s,t)} \right|$.

Example Find region D' which corresponds to D under transformation $x = \sqrt{st}$, $y = \sqrt{s/t}$



① Find which points map to the corners

looking for (s,t) s.t. $x(s,t) = 1$ and $y(s,t) = 1/2$

$$x = \sqrt{st}, y = \sqrt{s/t} \quad \begin{matrix} \text{multiply} \\ \text{divide} \end{matrix} \quad \begin{cases} xy = s \\ \frac{x}{y} = t \end{cases} \quad \begin{matrix} \text{This defines} \\ \text{the function} \\ \text{of } x \text{ only.} \end{matrix}$$

$$\text{Point 1: } (x,y) = (1, 1/2) \rightarrow s = 1 \cdot 1/2 = 1/2, t = (1/2)^2 = 1/4$$

$$\text{Point 2: } (x,y) = (2, 1) \rightarrow s = 2 \cdot 1 = 2, t = 1^2 = 1$$

and so on...

② Write equations in terms of s,t using substitution.

$$y = \frac{x}{2} \rightarrow \sqrt{s/t} = \frac{\sqrt{st}}{2} \rightarrow 2 = t$$

$$y = \frac{2}{x} \rightarrow \sqrt{s/t} = \frac{2}{\sqrt{st}} \rightarrow s = 2$$

$$y = 2x \rightarrow \sqrt{s/t} = 2\sqrt{st} \rightarrow t = 1/2$$

$$y = \frac{1}{2x} \rightarrow \sqrt{s/t} = \frac{1}{2\sqrt{st}} \rightarrow s = \frac{1}{2}$$

③ Compute the Jacobian $x = (st)^{1/2}, y = (s/t)^{1/2}$

$$\frac{\partial x}{\partial s} = \frac{1}{2}(st)^{-1/2} \cdot t \quad \frac{\partial y}{\partial s} = \frac{1}{2}(s/t)^{-1/2} \cdot \frac{1}{t}$$

$$\frac{\partial x}{\partial t} = \frac{1}{2}(st)^{-1/2} \cdot s \quad \frac{\partial y}{\partial t} = \frac{1}{2}(s/t)^{-1/2} \cdot \left(-\frac{s}{t^2}\right)$$

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} \frac{t}{2\sqrt{st}} & \frac{1}{2t\sqrt{s}} \\ \frac{s}{2\sqrt{st}} & -\frac{s}{2t^2\sqrt{s}} \end{vmatrix}$$

$$= -\frac{s}{4t\sqrt{s^3}} - \frac{s}{4t\sqrt{s^3}}$$

$$= -\frac{1}{2t} \quad \text{★ does not preserve orientation ★}$$

④ Compute the area of D'

$$A(D) = \iint_D 1 dA = \iint_{D'} \left| -\frac{1}{2t} \right| ds dt$$

$$= \int_{1/2}^2 \int_{1/2}^2 \frac{1}{2t} ds dt$$

$$= \int_{1/2}^2 \left[\frac{s}{2t} \right]_{1/2}^2 dt$$

$$= \int_{1/2}^2 \left(\frac{1}{t} - \frac{1}{4t} \right) dt$$

$$= \left[\ln(t) - \frac{1}{4} \ln(t) \right]_{1/2}^2 = \frac{3}{4} \ln(2) - \frac{3}{4} \ln(1/2) = \frac{3}{2} \ln(2)$$

Activity 11.9.3

Complete and discuss w/ the person next to you.

Class discussion.

Activity 11.9.3. Find the Jacobian when changing from rectangular to polar coordinates. That is, for the transformation given by $x = r \cos(\theta)$, $y = r \sin(\theta)$, determine a simplified expression for the quantity

$$\frac{\partial(x,y)}{\partial(r,\theta)}$$

What do you observe about your result? How is this connected to our earlier work with double integrals in polar coordinates?

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r$$

It's the same factor that showed up in the polar integral formula!

Activity 11.9.5

Complete and discuss w/ the person next to you.

Class discussion.

(a) $x+y=1$

(b) $\sqrt{x+y} (x-y)^2 = \sqrt{s} t^2$

(c) $y = s-x = s-(t+y) \Rightarrow y = \frac{s-t}{2}$

$$x = s-y = s-(x-t) \Rightarrow x = \frac{s+t}{2}$$

(d) (i) $1 = x+y = \frac{s+t}{2} + \frac{s-t}{2} = \frac{2s}{2} = s$

$$s=1$$

$$(ii) 0 = x = \frac{s+t}{2} \Rightarrow s = -t$$

$$(iii) 0 = y = \frac{s-t}{2} \Rightarrow s = t$$

$$\left| \frac{\partial(x,y)}{\partial(s,t)} \right| = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \frac{1}{2}$$

(e) $\iint_D \sqrt{x+y} (x-y)^2 dA = \iint_{D'} \sqrt{s} t^2 \left| \frac{\partial(x,y)}{\partial(s,t)} \right| ds dt$

$$= \int_0^1 \int_0^s \sqrt{s} t^2 \left| -\frac{1}{2} \right| dt ds$$

$$= \frac{1}{2} \int_0^1 \left[\frac{1}{3} \sqrt{s} t^3 \right]_0^s ds$$

$$= \frac{1}{6} \int_0^1 \sqrt{s} s^3 ds = \frac{1}{6} \int_0^1 s^{7/2} ds$$

$$= \frac{1}{6} \left[\frac{2}{9} s^{9/2} \right]_0^1 = \frac{2}{27}$$

Section 11.9.3

Change of Variables for Triple Integrals

Change of Variables Formula for Triple Integral.

Suppose $x = x(s,t,u)$, $y = y(s,t,u)$, $z = z(s,t,u)$ transforms a closed and bounded solid S in stu -coord into a closed and bounded solid S' in xyz -coordinates. Then

$$\iiint_S f(x,y,z) dV = \iiint_{S'} f(x(s,t,u), y(s,t,u), z(s,t,u)) \left| \frac{\partial(x,y,z)}{\partial(s,t,u)} \right| ds dt du$$

where $\frac{\partial(x,y,z)}{\partial(s,t,u)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} & \frac{\partial z}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \end{vmatrix}$ is the determinant of the Jacobian.

The determinant $\frac{\partial(x,y,z)}{\partial(s,t,u)}$ of the Jacobian is the factor by which the transformation expands or contracts the volume of a (small) region.

Since the transformation may not preserve orientation, we have to use the absolute value $\left| \frac{\partial(x,y,z)}{\partial(s,t,u)} \right|$.

Example

Activity 11.9.6. Consider the solid S defined by the inequalities $0 \leq x \leq 2$, $\frac{x}{2} \leq y \leq \frac{x}{2} + 1$, and $0 \leq z \leq 6$. Consider the transformation defined by $s = \frac{x}{2}$, $t = \frac{y-x}{2}$, and $u = \frac{z}{3}$. Let $f(x,y,z) = x - 2y + z$.

a. The transformation turns the solid S into a box S' in stu -coordinates. Apply the transformation to the boundaries of the solid S to find stu -coordinate descriptions of the box S' .

b. Find the Jacobian $\frac{\partial(x,y,z)}{\partial(s,t,u)}$.

c. Use the Jacobian to perform a change of variables and evaluate $\iiint_S f(x,y,z) dV$ by evaluating $\iiint_{S'} f(x(s,t,u), y(s,t,u), z(s,t,u)) \left| \frac{\partial(x,y,z)}{\partial(s,t,u)} \right| ds dt du$.

(a) Substitute the equations $s = \frac{x}{2}$, $t = \frac{y-x}{2}$, $u = \frac{z}{3}$ into the inequalities.

$$\begin{aligned} 0 \leq x \leq 2 & \xrightarrow{s=\frac{x}{2}} 0 \leq s \leq 1 \\ 0 \leq z \leq 6 & \xrightarrow{u=\frac{z}{3}} 0 \leq u \leq 2 \\ \frac{x}{2} \leq y \leq \frac{x}{2} + 1 & \xrightarrow{t=\frac{y-x}{2}} \frac{x-2(\frac{x}{2})}{2} \leq t \leq \frac{x-2(\frac{x}{2}+1)}{2} \end{aligned}$$

$$0 \leq t \leq \frac{1}{2}$$

$$(b) \frac{\partial(x,y,z)}{\partial(s,t,u)} = \begin{vmatrix} 2 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{vmatrix} = -6$$

$$(c) \iiint_S x - 2y + z dV = 6 \iiint_{S'} 2s - 2(s-t) + 3u ds dt du$$

$$= 6 \int_0^1 \int_0^{1/2} \int_0^2 2t + 3u ds dt du$$

$$= 6 \int_0^1 \int_0^{1/2} 2t + 3u dt du$$

$$= 6 \int_0^1 \left[\frac{1}{2} (2t + 3u)^2 \right]_0^{1/2} du$$

$$= 6 \int_0^1 \left(\frac{1}{2} (2 + 3u)^2 - \frac{1}{2} (3u)^2 \right) du$$

$$= 6 \int_0^1 (2 + 3u) du = 6 \left[2u + \frac{3}{2} u^2 \right]_0^1 = 6 \left(2 + \frac{3}{2} \right) = 15$$

$$= 15$$

$$\dots \text{ and so on.}$$